電気のための微分積分C
第3回解答

基本事項

[1] $\int f(x)\,dx = F(x) \quad \Rightarrow \quad \int_a^b f(x)\,dx = F(b) - F(a)$ $\Big($ これを $= \Big[F(x)\Big]_a^b$ と書く $\Big)$

[2] $\int_a^b f'(x)g(x)\,dx = \Big[f(x)g(x)\Big]_a^b - \int_a^b f(x)g'(x)\,dx$

[3] $x = \varphi(t),\ \varphi(\alpha) = a, \varphi(\beta) = b \ \Rightarrow \ \int_a^b f(x)\,dx = \int_\alpha^\beta f(\varphi(t))\varphi'(t)\,dt$

3.1. 次の定積分を計算せよ。

(1) $\int_1^2 x\sqrt{x-1}\,dx$

$x - 1 = t$ とおくと $\dfrac{dt}{dx} = \dfrac{d}{dx}(x-1) = 1$ だから $dx = dt$.

また, $x = t + 1$.

だから

$$\int x\sqrt{x-1}\,dx = \int (t+1)\sqrt{t}\,dt = \int t\sqrt{t}\,dt + \int \sqrt{t}\,dt$$
$$= \frac{2}{5}t^{\frac{5}{2}} + \frac{2}{3}t^{\frac{3}{2}} = \frac{2}{5}(x-1)^{\frac{5}{2}} + \frac{2}{3}(x-1)^{\frac{3}{2}}$$

[1] より

$$\int_1^2 x\sqrt{x-1}\,dx = \left[\frac{2}{5}(x-1)^{\frac{5}{2}} + \frac{2}{3}(x-1)^{\frac{3}{2}}\right]_1^2$$
$$= \left(\frac{2}{5}(2-1)^{\frac{5}{2}} + \frac{2}{3}(2-1)^{\frac{3}{2}}\right) - \left(\frac{2}{5}(1-1)^{\frac{5}{2}} + \frac{2}{3}(1-1)^{\frac{3}{2}}\right)$$
$$= \frac{2}{5} + \frac{2}{3} = \frac{16}{15}$$

[別解 1] $x - 1 = t$ とおき [3] を使うと

$x = 1 \Rightarrow t = 0, \quad x = 2 \Rightarrow t = 1$

だから

$$\int_1^2 x\sqrt{x-1}\,dx = \int_0^1 (t+1)\sqrt{t}\,dt = \left[\frac{2}{5}t^{\frac{5}{2}} + \frac{2}{3}t^{\frac{3}{2}}\right]_0^1 = \frac{2}{5} + \frac{2}{3} = \frac{16}{15}$$

[別解 2]

$$\int \sqrt{x-1}\,dx = \frac{2}{3}(x-1)^{\frac{3}{2}}$$

だから [2] より

$$\int_1^2 x\sqrt{x-1}\,dx = \int_1^2 x\left(\frac{2}{3}(x-1)^{\frac{3}{2}}\right)'\,dx$$

$$= \left[x\left(\frac{2}{3}(x-1)^{\frac{3}{2}}\right)\right]_1^2 - \int_1^2 \left(\frac{2}{3}(x-1)^{\frac{3}{2}}\right)\,dx$$

$$= \left[x\left(\frac{2}{3}(x-1)^{\frac{3}{2}}\right)\right]_1^2 - \left[\frac{2}{3}\times\frac{2}{5}(x-1)^{\frac{5}{2}}\right]_1^2$$

$$= \frac{4}{3} - \frac{4}{15} = \frac{16}{15}$$

(2) $\displaystyle\int_1^5 x\sqrt{2x-1}\,dx$

$2x-1=t$ とおくと $\dfrac{dt}{dx} = \dfrac{d}{dx}(2x-1) = 2$ だから $dx = \dfrac{dt}{2}$.

また, $x = \dfrac{t+1}{2}$.

だから

$$\int x\sqrt{2x-1}\,dx = \int \frac{t+1}{2}\sqrt{t}\,\frac{dt}{2} = \frac{1}{4}\int t\sqrt{t}\,dt + \frac{1}{4}\int \sqrt{t}\,dt$$

$$= \frac{1}{4}\int t^{\frac{3}{2}}\,dt + \frac{1}{4}\int t^{\frac{1}{2}}\,dt = \frac{1}{4}\frac{2}{5}t^{\frac{5}{2}} + \frac{1}{4}\frac{2}{3}t^{\frac{3}{2}}$$

x が 1 から 5 まで動くとき t は 1 から 9 まで動くから

$$\int_1^5 x\sqrt{2x-1}\,dx = \frac{1}{4}\int_1^9 (t+1)\sqrt{t}\,dt = \left[\frac{1}{4}\frac{2}{5}t^{\frac{5}{2}} + \frac{1}{4}\frac{2}{3}t^{\frac{3}{2}}\right]_1^9 = \frac{428}{15}$$

(3) $\displaystyle\int_0^{\frac{\pi}{2}} x\sin x\,dx$

$$\int \sin x\,dx = -\cos x$$

だから [2] を使って

$$\int_0^{\frac{\pi}{2}} x\sin x\,dx = \int_0^{\frac{\pi}{2}} x\left(-\cos x\right)'\,dx = \left[x\left(-\cos x\right)\right]_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} \left(-\cos x\right)\,dx$$
$$= \left[x\left(-\cos x\right)\right]_0^{\frac{\pi}{2}} - \left[\left(-\sin x\right)\right]_0^{\frac{\pi}{2}} = 1$$

(4) $\displaystyle\int_0^{\frac{\pi}{2}} x\sin 2x\,dx$

$\displaystyle\int \sin 2x\,dx = -\frac{1}{2}\cos 2x$ であるから $\displaystyle\left(-\frac{1}{2}\cos 2x\right)' = \sin 2x$ したがって

$$\int_0^{\frac{\pi}{2}} x\sin 2x\,dx = \int_0^{\frac{\pi}{2}} x\left(-\frac{1}{2}\cos 2x\right)'\,dx$$
$$= \left[x\left(-\frac{1}{2}\cos 2x\right)\right]_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} (x)'\left(-\frac{1}{2}\cos 2x\right)\,dx$$

$\displaystyle\int \cos 2x\,dx = \frac{1}{2}\sin 2x$ だから

$$= \frac{\pi}{2}\left(-\frac{1}{2}\cos\pi\right) + \left[\frac{1}{4}\sin 2x\right]_0^{\frac{\pi}{2}}$$
$$= \frac{\pi}{4}$$

(5) $\displaystyle\int_0^1 xe^{-x}, dx$

$$\int e^{-x}\,dx = -e^{-x}$$

だから [2] を使って

$$\int_0^1 xe^{-x}\,dx = \int_0^1 x\left(-e^{-x}\right)'\,dx = \left[x\left(-e^{-x}\right)\right]_0^1 - \int_0^1 \left(-e^{-x}\right)\,dx$$
$$= \left[x\left(-e^{-x}\right)\right]_0^1 - \left[\left(e^{-x}\right)\right]_0^1 = 1 - 2e^{-1}$$

(6) $\displaystyle\int_0^1 x(2x-1)^4\,dx$

$2x-1=t$ とおくと $\dfrac{dt}{dx}=\dfrac{d}{dt}(2x-1)=2$ だから $dx=\dfrac{dt}{2}$.
また,
$x=\dfrac{t+1}{2},\quad x=0\Rightarrow t=-1,\quad x=1\Rightarrow t=1$
だから
$$\int_0^1 x(2x-1)^4\,dx=\int_{-1}^1\frac{t+1}{2}t^4\,\frac{dt}{2}=\left[\frac{1}{24}t^6+\frac{1}{20}t^5\right]_{-1}^1=\frac{1}{10}$$

(7) $\displaystyle\int_0^1 x(x^2-1)^5\,dx$

$x^2-1=t$ とおくと $\dfrac{dt}{dx}=\dfrac{d}{dt}(x^2-1)=2x$ だから $dx=\dfrac{dt}{2x}$.
また,
$x=0\Rightarrow t=-1,\quad x=1\Rightarrow t=0$
だから
$$\int_0^1 x(x^2-1)^5\,dx=\int_{-1}^0 xt^5\,\frac{dt}{2x}=\left[\frac{1}{12}t^6\right]_{-1}^0=0-\frac{1}{12}(-1)^6=-\frac{1}{12}$$

(8) $\displaystyle\int_0^\pi e^x\,\cos x\,dx=I$ とおくと

$$\begin{aligned}I&=\Big[e^x\,\sin x\Big]_0^\pi-\int_0^\pi e^x\,\sin x\,dx\\&=\Big[e^x\,\sin x\Big]_0^\pi-\Big[e^x\,(-\cos x)\Big]_0^\pi+\int_0^\pi e^x\,(-\cos x)\,dx\\&=\Big[e^x\,\sin x\Big]_0^\pi-\Big[e^x\,(-\cos x)\Big]_0^\pi-I=-e^\pi-1-I\end{aligned}$$

だから $I=\dfrac{-1-e^\pi}{2}$.
$e^x\cos x=\mathrm{Re}(e^{(1+i)x})$ とみて複素指数関数の積分を用いて

$$I = \mathrm{Re}\left(\int_0^\pi e^{(1+i)x}\,dx\right) = \mathrm{Re}\left(\left[\frac{e^{(1+i)x}}{1+i}\right]_0^\pi\right) = \mathrm{Re}\left(\left[\frac{(1-i)e^{(1+i)x}}{2}\right]_0^\pi\right)$$
$$= \left[\mathrm{Re}\left(\frac{(1-i)e^x(\cos x + i\sin x)}{2}\right)\right]_0^\pi = \left[\frac{e^x(\cos x + \sin x)}{2}\right]_0^\pi$$
$$= \frac{e^\pi \cos\pi - e^0\cos 0)}{2} = \frac{-e^\pi - 1}{2}$$

としてやってもよい。

(9) $\sin x = t$ とおくと

$\dfrac{dt}{dx} = \cos x$ だから $dx = \dfrac{dt}{\cos x}$

$\cos^3 x = (1-\sin^2 x)\cos x = (1-t^2)\cos x$

x が 0 から $\dfrac{\pi}{2}$ まで動くと t は 0 から 1 まで動く

ので

$$\int_0^{\frac{\pi}{2}} \cos^3 x\,dx = \int_0^1 (1-t^2)\,dt = \left[t - \frac{t^3}{3}\right]_0^1 = \frac{2}{3}$$

(10) $\tan\dfrac{x}{2} = t$ とおくと

$\sin x = \dfrac{2t}{1+t^2}$,

$dx = \dfrac{2dt}{1+t^2}$,

x が 0 から $\dfrac{\pi}{2}$ まで動くとき t は 0 から 1 まで動く

となるので置換積分法により

$$\int_0^{\frac{\pi}{2}} \frac{1}{1+\sin x}\,dx = \int_0^1 \frac{1}{1+\frac{2t}{1+t^2}}\,\frac{2\,dt}{1+t^2} = \int_0^1 \frac{2\,dt}{1+2t+t^2}$$
$$= \int_0^1 2(1+t)^{-2}\,dt = [-2(1+t)^{-1}]_0^1 = 1.$$