

電気のための微分積分C 第3回解答

基本事項

$$[1] \int f(x) dx = F(x) \Rightarrow \int_a^b f(x) dx = F(b) - F(a) \quad \left(\text{これを} = \left[F(x) \right]_a^b \text{と書く} \right)$$

$$[2] \int_a^b f'(x)g(x) dx = \left[f(x)g(x) \right]_a^b - \int_a^b f(x)g'(x) dx$$

$$[3] x = \varphi(t), \varphi(\alpha) = a, \varphi(\beta) = b \Rightarrow \int_a^b f(x) dx = \int_\alpha^\beta f(\varphi(t))\varphi'(t) dt$$

3.1. 次の定積分を計算せよ。

$$(1) \int_1^2 x\sqrt{x-1} dx$$

$$x-1=t \text{ とおくと } \frac{dt}{dx} = \frac{d}{dx}(x-1) = 1 \text{ だから } dx = dt.$$

$$\text{また, } x = t + 1.$$

だから

$$\begin{aligned} \int x\sqrt{x-1} dx &= \int (t+1)\sqrt{t} dt = \int t\sqrt{t} dt + \int \sqrt{t} dt \\ &= \frac{2}{5}t^{\frac{5}{2}} + \frac{2}{3}t^{\frac{3}{2}} = \frac{2}{5}(x-1)^{\frac{5}{2}} + \frac{2}{3}(x-1)^{\frac{3}{2}} \end{aligned}$$

[1] より

$$\begin{aligned} \int_1^2 x\sqrt{x-1} dx &= \left[\frac{2}{5}(x-1)^{\frac{5}{2}} + \frac{2}{3}(x-1)^{\frac{3}{2}} \right]_1^2 \\ &= \left(\frac{2}{5}(2-1)^{\frac{5}{2}} + \frac{2}{3}(2-1)^{\frac{3}{2}} \right) - \left(\frac{2}{5}(1-1)^{\frac{5}{2}} + \frac{2}{3}(1-1)^{\frac{3}{2}} \right) \\ &= \frac{2}{5} + \frac{2}{3} = \frac{16}{15} \end{aligned}$$

[別解 1] $x-1=t$ とおき [3] を使うと

$$x=1 \Rightarrow t=0, \quad x=2 \Rightarrow t=1$$

だから

$$\int_1^2 x\sqrt{x-1} dx = \int_0^1 (t+1)\sqrt{t} dt = \left[\frac{2}{5}t^{\frac{5}{2}} + \frac{2}{3}t^{\frac{3}{2}} \right]_0^1 = \frac{2}{5} + \frac{2}{3} = \frac{16}{15}$$

[別解 2]

$$\int \sqrt{x-1} dx = \frac{2}{3}(x-1)^{\frac{3}{2}}$$

だから [2] より

$$\begin{aligned} \int_1^2 x\sqrt{x-1} dx &= \int_1^2 x \left(\frac{2}{3}(x-1)^{\frac{3}{2}} \right)' dx \\ &= \left[x \left(\frac{2}{3}(x-1)^{\frac{3}{2}} \right) \right]_1^2 - \int_1^2 \left(\frac{2}{3}(x-1)^{\frac{3}{2}} \right) dx \\ &= \left[x \left(\frac{2}{3}(x-1)^{\frac{3}{2}} \right) \right]_1^2 - \left[\frac{2}{3} \times \frac{2}{5}(x-1)^{\frac{5}{2}} \right]_1^2 \\ &= \frac{4}{3} - \frac{4}{15} = \frac{16}{15} \end{aligned}$$

$$(2) \int_0^{\frac{\pi}{2}} x \sin x dx$$

$$\int \sin x dx = -\cos x$$

だから [2] を使って

$$\begin{aligned} \int_0^{\frac{\pi}{2}} x \sin x dx &= \int_0^{\frac{\pi}{2}} x (-\cos x)' dx = [x(-\cos x)]_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} (-\cos x) dx \\ &= [x(-\cos x)]_0^{\frac{\pi}{2}} - [(-\sin x)]_0^{\frac{\pi}{2}} = 1 \end{aligned}$$

$$(3) \int_0^1 xe^{-x}, dx$$

$$\int e^{-x} dx = -e^{-x}$$

だから [2] を使って

$$\begin{aligned} \int_0^1 xe^{-x} dx &= \int_0^1 x (-e^{-x})' dx = [x(-e^{-x})]_0^1 - \int_0^1 (-e^{-x}) dx \\ &= [x(-e^{-x})]_0^1 - [(e^{-x})]_0^1 = 1 - 2e^{-1} \end{aligned}$$

$$(4) \int_0^1 x(2x-1)^4 dx$$

$$2x-1=t \text{ とおくと } \frac{dt}{dx} = \frac{d}{dt}(2x-1) = 2 \text{ だから } dx = \frac{dt}{2}.$$

また,

$$x = \frac{t+1}{2}, \quad x=0 \Rightarrow t=-1, \quad x=1 \Rightarrow t=1$$

だから

$$\int_0^1 x(2x-1)^4 dx = \int_{-1}^1 \frac{t+1}{2} t^4 \frac{dt}{2} = \left[\frac{1}{24} t^6 + \frac{1}{20} t^5 \right]_{-1}^1 = \frac{1}{10}$$

$$(5) \int_0^1 x(x^2-1)^5 dx$$

$$x^2-1=t \text{ とおくと } \frac{dt}{dx} = \frac{d}{dt}(x^2-1) = 2x \text{ だから } dx = \frac{dt}{2x}.$$

また,

$$x=0 \Rightarrow t=-1, \quad x=1 \Rightarrow t=0$$

だから

$$\int_0^1 x(x^2-1)^5 dx = \int_{-1}^0 x t^5 \frac{dt}{2x} = \left[\frac{1}{12} t^6 \right]_{-1}^0 = 0 - \frac{1}{12} (-1)^6 = -\frac{1}{12}$$

$$(6) \int_0^\pi e^x \cos x dx = I \text{ とおくと}$$

$$\begin{aligned} I &= \left[e^x \sin x \right]_0^\pi - \int_0^\pi e^x \sin x dx \\ &= \left[e^x \sin x \right]_0^\pi - \left[e^x (-\cos x) \right]_0^\pi + \int_0^\pi e^x (-\cos x) dx \\ &= \left[e^x \sin x \right]_0^\pi - \left[e^x (-\cos x) \right]_0^\pi - I = -e^\pi - 1 - I \end{aligned}$$

$$\text{だから } I = \frac{-1 - e^\pi}{2}.$$

$e^x \cos x = \operatorname{Re}(e^{(1+i)x})$ とみて複素指数関数の積分を用いて

$$\begin{aligned}
I &= \operatorname{Re} \left(\int_0^\pi e^{(1+i)x} dx \right) = \operatorname{Re} \left(\left[\frac{e^{(1+i)x}}{1+i} \right]_0^\pi \right) = \operatorname{Re} \left(\left[\frac{(1-i)e^{(1+i)x}}{2} \right]_0^\pi \right) \\
&= \operatorname{Re} \left(\frac{(1-i)(-e^\pi - 1)}{2} \right) = \frac{-e^\pi - 1}{2}
\end{aligned}$$

としてやってもよい。

(7) $\sin x = t$ とおくと

$$\frac{dt}{dx} = \cos x \text{ だから } dx = \frac{dt}{\cos x}$$

$$\cos^3 x = (1 - \sin^2 x) \cos x = (1 - t^2) \cos x$$

x が 0 から $\frac{\pi}{2}$ まで動くとき t は 0 から 1 まで動く

ので

$$\int_0^{\frac{\pi}{2}} \cos^3 x dx = \int_0^1 (1 - t^2) dt = \left[t - \frac{t^3}{3} \right]_0^1 = \frac{2}{3}$$

(8) $\tan \frac{x}{2} = t$ とおくと

$$\sin x = \frac{2t}{1+t^2},$$

$$dx = \frac{2dt}{1+t^2},$$

x が 0 から $\frac{\pi}{2}$ まで動くとき t は 0 から 1 まで動く

となるので置換積分法により

$$\begin{aligned}
\int_0^{\frac{\pi}{2}} \frac{1}{1 + \sin x} dx &= \int_0^1 \frac{1}{1 + \frac{2t}{1+t^2}} \frac{2 dt}{1+t^2} = \int_0^1 \frac{2 dt}{1 + 2t + t^2} \\
&= \int_0^1 2(1+t)^{-2} dt = [-2(1+t)^{-1}]_0^1 = 1.
\end{aligned}$$

追加

$$(9) \int_1^5 x \sqrt{2x-1} dx$$

$2x - 1 = t$ とおくと $\frac{dt}{dx} = \frac{d}{dx}(2x - 1) = 2$ だから $dx = \frac{dt}{2}$.

また, $x = \frac{t+1}{2}$.

だから

$$\begin{aligned}\int x\sqrt{2x-1} dx &= \int \frac{t+1}{2} \sqrt{t} \frac{dt}{2} = \frac{1}{4} \int t\sqrt{t} dt + \frac{1}{4} \int \sqrt{t} dt \\ &= \frac{1}{4} \int t^{\frac{3}{2}} dt + \frac{1}{4} \int t^{\frac{1}{2}} dt = \frac{1}{4} \frac{2}{5} t^{\frac{5}{2}} + \frac{1}{4} \frac{2}{3} t^{\frac{3}{2}}\end{aligned}$$

x が 1 から 5 まで動くとき t は 1 から 9 まで動くから

$$\int_1^5 x\sqrt{2x-1} dx = \frac{1}{4} \int_1^9 (t+1)\sqrt{t} dt = \left[\frac{1}{4} \frac{2}{5} t^{\frac{5}{2}} + \frac{1}{4} \frac{2}{3} t^{\frac{3}{2}} \right]_1^9 = \frac{428}{15}$$

$$(10) \int_0^{\frac{\pi}{2}} x \sin 2x dx$$

$\int \sin 2x dx = -\frac{1}{2} \cos 2x$ であるから $\left(-\frac{1}{2} \cos 2x\right)' = \sin 2x$ したがって

$$\begin{aligned}\int_0^{\frac{\pi}{2}} x \sin 2x dx &= \int_0^{\frac{\pi}{2}} x \left(-\frac{1}{2} \cos 2x\right)' dx \\ &= \left[x \left(-\frac{1}{2} \cos 2x\right) \right]_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} (x)' \left(-\frac{1}{2} \cos 2x\right) dx \\ &= \frac{\pi}{2} \left(-\frac{1}{2} \cos \pi\right) - \left[\frac{1}{4} \sin 2x \right]_0^{\frac{\pi}{2}} \\ &= \frac{\pi}{4}\end{aligned}$$